# Parametric Higher-Order Abstract Syntax for Mechanized Semantics

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# The Big Picture

- We want to write programs that manipulate syntax including *variable binders*...
- ...in the language of a proof assistant, which enforces termination of all functions...
- ...and then we want to build short, mechanized proofs that we did it right.

# Running Example: CPS Translation

**Source Language:**

$\tau$ ::= **bool** | $\tau \to \tau$

e ::= $x$ | **true** | **false** | e e | $\lambda x : \tau$. e

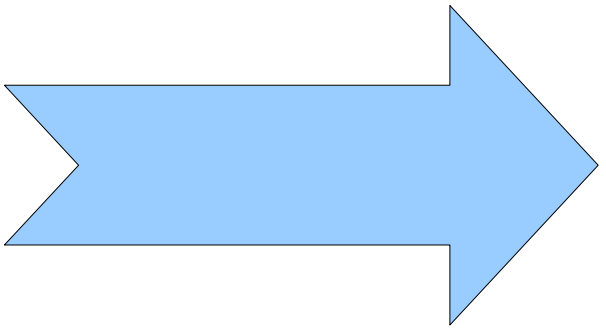
**Target Language:**

$\tau$ ::= **bool** | $\tau \to 0$ | $\tau \times \tau$

p ::= **true** | **false** | $\lambda x : \tau$. e | $(x, x)$ | $x.1$ | $x.2$

e ::= $x\ x$ | **let** $x$ = p **in** e

$(\lambda x$ : **bool**. $x)$ **true**



**let** $f = \lambda p.$
  **let** $x = p.1$ **in**
  **let** k $= p.2$ **in**
  $k\ x$ **in**
**let** $t$ = **true in**
**let** $p = (t, k_{\textbf{top}})$ **in**
$f\ p$

# An Embarrassment of Riches

Choices for representing binders:

- Concrete syntax
- De Bruijn indices/levels
- Locally nameless
- Nominal logic
- Higher-order abstract syntax
- ...

- Why not add one more? :-)

# Concrete Syntax

[Church 1936?]

```
type var = string

type typ =
    | Bool
    | Arrow of typ * typ

type exp =
    | Var of var
    | True
    | False
    | App of exp * exp
    | Abs of var * exp
```

# Concrete CPS Translation

```
let rec cpsExp (e : exp) (k : var -> cexp) : cexp =
    match e with
      | Var x -> k x
      | True -> withFresh (fun x -> Let x CTrue (k x))
      | False -> withFresh (fun x -> Let x CFalse (k x))
      | App e1 e2 -> cpsExp e1 (fun  f : var ->
          cpsExp e2 (fun x ->
              withFresh (fun k' ->
                  Let k' (withFresh (fun a -> CAbs a (k a)))
                  (withFresh (fun p ->
                      Let p (Pair (x, k')) (Call f p)))))
      | Abs x e' -> withFresh (fun f ->
          Let f (withFresh (fun p -> CAbs p (
              Let x (Fst p)
                  (withFresh (fun k' -> Let k' (Snd p)
                      cpsExp e' (fun r -> Call k' r)))))
          (k f))
```

# De Bruijn CPS Translation

```
let rec cpsExp (e : exp) (k : int -> var -> cexp) : cexp =
    match e with
        | Var x -> k 0 x
        | True -> Let CTrue (k 1 0)
        | False -> Let CFalse (k 1 0)
        | App e1 e2 -> cpsExp e1 (fun (nf : int) (f : var) ->
            cpsExp e2 (fun nx x ->
                Let (CAbs (k (nf + nx) 0)))
                    (Let (Pair (x + 1, 0))
                        (Call (f + nx + 2, 0)))))
        | Abs e' -> Let (CAbs (
                Let (Snd 0)
                    (Let (Fst 1)
                        cpsExp e' (fun nr r ->
                            Call (nr + 1) r))))
            (k 1 0)
```

# Higher-Order Abstract Syntax

[Church 1940]

```
type exp =
    | True
    | False
    | App of exp * exp
    | Abs of exp -> exp
```

$\lambda x, \lambda y, x$  Abs (fun *x* -> Abs (fun _ -> *x*))

```
let maybeBeta e =
    match e with
        | App (Abs f) e' -> f e'
        | _ -> e
```

# Pandora's Box

```
let f e =
    match e with
        | Abs e' -> e' e
        | _ -> e
```

```
type exp =
    | True
    | False
    | App of exp * exp
    | Abs of exp -> exp
```

f (Abs f)

- ➔ **match** Abs f **with** Abs e' -> e' (Abs f) | _ -> Abs f
- ➔ f (Abs f)
- ➔ **match** Abs f **with** Abs e' -> e' (Abs f) | _ -> Abs f
- ➔ f (Abs f)
- ➔ ....

# Weak HOAS

[Despeyroux et al. 1995, Honsell et al. 2001]

```
type var      (* Abstract type! *)

type exp =
    | Var of var
    | True
    | False
    | App of exp * exp
    | Abs of var -> exp
```

$\lambda x, \lambda y, x$  Abs (fun *x* -> Abs (fun _ -> Var *x*))

# Getting Dependent

```
type 't var

type 't exp =
    | Var : 't var -> 't exp
    | True : bool exp
    | False : bool exp
    | App : ('d -> 'r) exp * 'd exp -> 'r exp
    | Abs : ('d var -> 'r exp) -> ('d -> 'r) exp
```

# Getting Dependent

~~**type** '*t* var~~

**type** ('*t*, '*V*) exp =
    | Var : '*t* '*V* -> ('*t*, '*V*) exp
    | True : (bool, '*V*) exp
    | False : (bool, '*V*) exp
    | App : ('*d* -> '*r*, '*V*) exp * ('*d*, '*V*) exp -> ('*r*, '*V*) exp
    | Abs : ('*d* '*V* -> ('*r*, '*V*) exp, '*V*) -> ('*d* -> '*r*, '*V*) exp

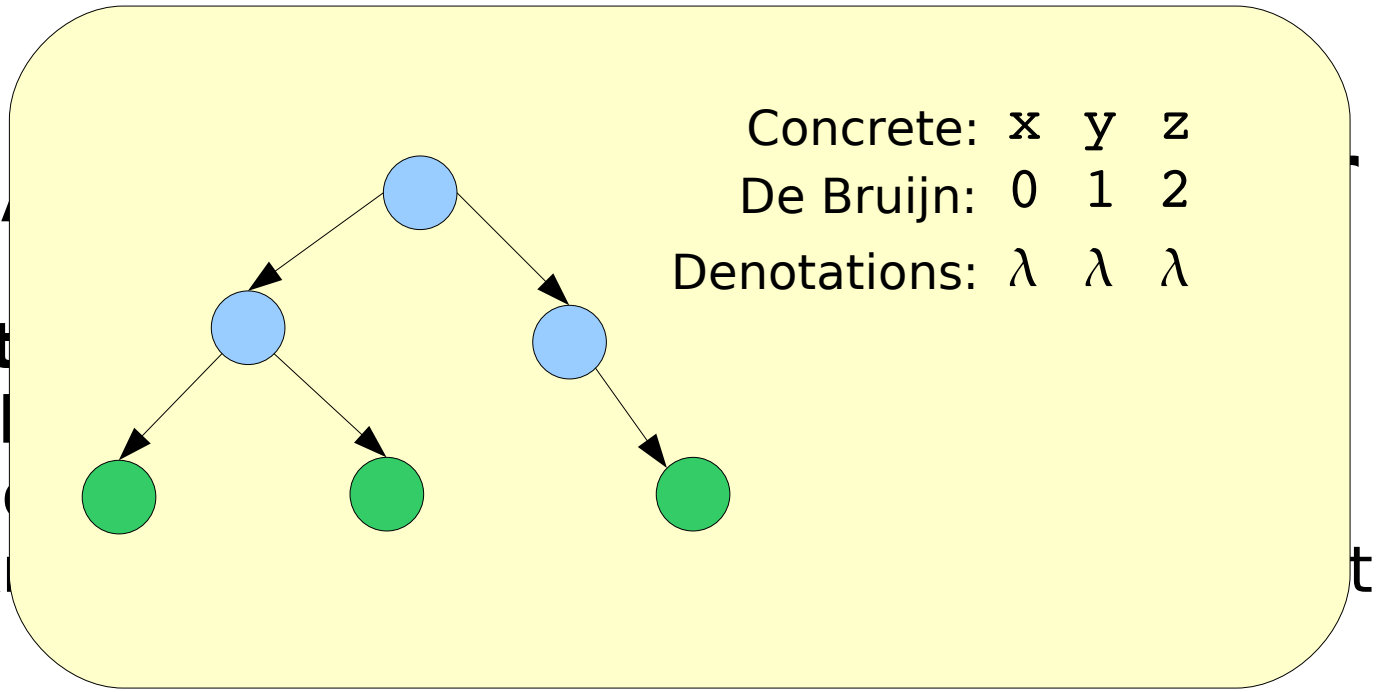
**type** '*t* Exp = **forall** '*V*. '*t* exp('*V*)
(* [Boxes Go Bananas, Washburn and Weirich 2003] *)

(* Will call this representation scheme “Parametric HOAS,” or “PHOAS”.... *)

# PHOAS CPS Translation

```
let rec cpsExp (e : ('t, 'V o cpsTyp) exp)
        (k : (cpsTyp 't) 'V -> 'V cexp) : 'V cexp =
    match e with
      | Var x -> k x
      | True -> Let CTrue k
      | F
      | A                                              ->
                                                  Call f p))))
      | Abs e' -> Let (CAbs (fun p ->
          Let (Fst p) (fun x ->
              Let (Snd p) (fun k' ->
                  cpsExp (e' x) (fun r -> Call k' r))))) k
```

```
(Var x) : ('t, 'V o cpsTyp) exp
x : 't ('V o cpsTyp)
x : (cpsTyp 't) 'V
(k x) : 'V cexp
```




**Fixpoint**
  **matc**
    | B
    | A
  **end.**

te *t2*

**Fixpoint** expDenote (*t* : typ) (*e* : exp typDenote *t*)
    : typDenote *t* =
  **match** *e* **with**
    | Var *x* => *x*
    | True => true
    | False => false
    | App (*e1*, *e2*) => (expDenote *e1*) (expDenote *e2*)
    | Abs *e'* => **fun** *x* => expDenote (*e' x*)
  **end.**

# Total Correctness Theorem

**Theorem:** For any source term *E* of type bool,
CexpDenote (CpsExp *E* (**fun** *x* -> Halt *x*)) = ExpDenote *E*

**Axiom:** For any source term *E*
and variable types *U* and *V*,
(*E U*) and (*E V*) are syntactically equivalent,
up to replacement of *U*'s with *V*'s.

Informally, an easy consequence of a **parametricity** theorem....

# A Proof

```
Scheme pterm_mut := Induction for pterm Sort Prop
with pprimop_mut := Induction for pprimop Sort Prop.

Section splice_correct.
  Variables result1 result2 : ptype.

  Variable e2 : ptypeDenote result1
    -> pterm ptypeDenote result2.

  Theorem splice_correct : forall e1 k,
    ptermDenote (splice e1 e2) k
    = ptermDenote e1 (fun r => ptermDenote (e2 r) k).
    apply (pterm_mut
      (fun e1 => forall k,
        ptermDenote (splice e1 e2) k
        = ptermDenote e1 (fun r => ptermDenote (e2 r) k))
      (fun t p => forall k,
        pprimopDenote (splicePrim p e2) k
        = pprimopDenote p (fun r => ptermDenote (e2 r) k)));
    equation.
  Qed.
End splice_correct.

Fixpoint lr (t : type) : typeDenote t -> ptypeDenote (cpsType t)
    -> Prop :=
  match t
    return (typeDenote t -> ptypeDenote (cpsType t) -> Prop) with
    | TBool => fun n1 n2 => n1 = n2
    | TArrow t1 t2 => fun f1 f2 =>
      forall x1 x2, lr _ x1 x2
        -> forall k, exists r,
          f2 (x2, k) = k r
          /\ lr _ (f1 x1) r
  end.

Lemma cpsTerm_correct : forall G t (e1 : term _ t) (e2 : term _ t),
  term_equiv G e1 e2
  -> (forall t v1 v2, In (vars (v1, v2)) G -> lr t v1 v2)
  -> forall k, exists r,
    ptermDenote (cpsTerm e2) k = k r
    /\ lr t (termDenote e1) r.
  Hint Rewrite splice_correct : ltamer.

  Ltac my_changer :=
    match goal with
      [ H : (forall (t : _) (v1 : _) (v2 : _),
        vars _ = vars _ \/ In _ _ -> _) -> _ |- _ ] =>
      match typeof H with
        | ?P -> _ =>
          assert P; [intros; push_vars; intuition; fail 2
            | idtac]
      end
    end.

  Ltac my_simpler := repeat progress (equation;
      fold ptypeDenote in *;
    fold cpsType in *; try my_changer).

  Ltac my_chooser T k :=
    match T with
      | bool => fail 1
      | type => fail 1
      | ctxt _ _ => fail 1
      | _ => default_chooser T k
    end.

  induction 1; matching my_simpler my_chooser; eauto.
Qed.

Theorem CpsTerm_correct : forall t (E : Term t),
  forall k, exists r,
    PtermDenote (CpsTerm E) k = k r
    /\ lr t (TermDenote E) r.
  unfold PtermDenote, TermDenote, CpsTerm; simpl; intros.
  eapply (cpsTerm_correct (G := nil)); simpl; intuition.
  apply Term_equiv.
Qed.

Theorem CpsTerm_correct_bool : forall (E : Term TBool),
  forall k, PtermDenote (CpsTerm E) k = k (TermDenote E).
  intros.
  generalize (CpsTerm_correct E k); firstorder congruence.
Qed.
```

# Proof Size Comparison

| | | | |
|---|---|---|---|
| Minamide & Okuma 2003 | Isabelle/HOL | Concrete | ~600 |
| Tian 2006 | Twelf | HOAS | ~50 |
| Dargaye & Leroy 2007 | Coq | De Bruijn | ~1700 |
| This case study | Coq | PHOAS | ~30 |
| | | | |
| Expanded case study | Coq | PHOAS | ~150 |

# More PHOAS + Definitional Compilers

- Case studies in this paper:
  - Closure conversion for STLC
  - Compilation of ML-style pattern matching
  - CPS translation for System F
- In other ongoing work:
  - CPS translation for idealized Core ML

# Conclusion

**Lambda Tamer** library available on the Web at:
`http://ltamer.sourceforge.net/`

PHOAS and definitional compilers combine to enable mechanized correctness proofs that are even easier to construct and maintain than those in ICFP papers.